

ON A SELECTION PROBLEM FOR SMALL NOISE PERTURBATION IN MULTIDIMENSIONAL CASE

ANDREY PILIPENKO AND FRANK NORBERT PROSKE

ABSTRACT. The problem on identification of a limit of an ordinary differential equation with discontinuous drift that perturbed by a zero-noise is considered in multidimensional case. This problem is a classical subject of stochastic analysis, see, for example, [6, 27, 11, 20]. However the multidimensional case was poorly investigated. We assume that the drift coefficient has a jump discontinuity along a hyperplane and is Lipschitz continuous in the upper and lower half-spaces. It appears that the behavior of the limit process depends on signs of the normal component of the drift at the upper and lower half-spaces in a neighborhood of the hyperplane, all cases are considered.

1. INTRODUCTION

Consider the Cauchy problem

$$X_t = x + \int_0^t b(X_s) ds, t \geq 0 \quad (1.1)$$

for $x \in \mathbb{R}^d$, where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable vector field.

If b satisfies a Lipschitz and linear growth condition, it is well-known that there exists a global unique solution $X \in C([0, \infty); \mathbb{R}^d)$ to (1.1).

However, if b is not Lipschitzian, the situation may change dramatically and well-posedness of (1.1) in the sense of uniqueness or even existence of solutions may fail.

An example of such a function is given by

$$b(x) = 2\operatorname{sgn}(x)\sqrt{|x|} \quad (1.2)$$

for $X_0 = 0$ and dimension $d = 1$, where the extremal trajectories $X_t = +t^2, -t^2$ and the zero curve $X_t = 0, t \geq 0$ are solutions to (1.1) among infinitely other ones.

Another example in the case of a discontinuous vector field is

$$b(x) = \operatorname{sgn}(x) \quad (1.3)$$

for $X_0 = 0$ with infinitely many solutions, where $X_t = +t, -t, t \geq 0$ are extremal solutions.

If we merely require that the vector field b is continuous, satisfying a growth condition of the form $\langle b(x), x \rangle \leq K(|x|^2 + 1)$ then it follows from Peano's theorem and the theorem of Arzelà-Ascoli that the set $C(x)$ of solutions $X \in C([0, \infty); \mathbb{R}^d)$ of (1.1) is non-empty and compact in $C([0, \infty); \mathbb{R}^d)$. Moreover, $C(x)$ is connected. See [26].

Date: October 6, 2015.

2010 Mathematics Subject Classification. 60H10, 49N60.

Key words and phrases. Keywords.

Research is partially supported by FP7-People-2011-IRSES Project number 295164.

Here the problem of uniqueness of solutions of (1.1) for an initial distribution μ_0 on \mathbb{R}^d , which corresponds to the case, when $C(x)$ is a singleton for x μ_0 -a.e., can be characterized by unique solutions of narrowly mesurable families of probability measures $(\mu_t)_{t \geq 0}$ on $C([0, \infty); \mathbb{R}^d)$ satisfying the continuity equation

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(b \cdot \nabla f) ds, t \geq 0 \quad (1.4)$$

for all $f \in C_c^\infty(\mathbb{R}^d)$ (space of smooth functions with compact support). It turns out that such solutions have the representation $\mu_t = \pi_t \mu$, $t \geq 0$ for projections π_t and a probability measure μ on $C([0, \infty); \mathbb{R}^d)$ called a superposition of solutions of the ODE (1.1). See [13], [1] and [15] for more information on the concept of superposition of solutions.

Using the concept of renormalized solutions, we mention that for $b \in W^{1,1}(\mathbb{R}^d)$ with $\operatorname{div}(b) = 0$ and initial distributions μ_0 with $\frac{\partial \mu_0}{\partial x} \in L^\infty(\mathbb{R}^d)$ it can be shown that the continuity equation has a unique solution $(\mu_t)_{t \geq 0}$ in the subclass of solutions for which $\frac{\partial \mu_t}{\partial x} \in L^\infty(\mathbb{R}^d)$ for all $t \geq 0$. See [13], [1] and [2].

On the other hand, the case, when $C(x)$ is not a singleton, gives rise to the natural question of how an "appropriate" or "meaningful" solution to (1.1) can be selected.

One important method in connection with this selection problem is due to Krylov [21], who constructed Markov selections, that is families of superposition solutions $(\mu^x)_{x \in \mathbb{R}^d}$ such that

$$\mu_{t+s}^x = \int_{\mathbb{R}^d} \mu_s^x \mu_t^x(dy), t, s \geq 0, x \in \mathbb{R}^d$$

holds for $\mu_t^x = \pi_t \mu^x$.

Another crucial approach which we want to employ in this paper is that of zero-noise selection, that is the selection of a solution X_\cdot to (1.1) as a limiting value of solutions X_\cdot^ε of the ODE (1.1) perturbed by a small noise $\varepsilon w(\cdot)$ given by the stochastic differential equation (SDE)

$$X_t^\varepsilon = x + \int_0^t b(X_s) ds + \varepsilon w(t) \quad (1.5)$$

for $\varepsilon \searrow 0$ in the sense of convergence in law, where $w(\cdot)$ is a d -dimensional Wiener process. The motivation for this selection principle comes from the desire to construct solutions to (1.1), which are stable under random perturbations.

The first results in this direction, that is when $C(x)$ is not a singleton ("Peano phenomenon"), was obtained in the foundational papers of Bafico [5] and Bafico, Baldi [6] in the case of one-dimensional time-homogeneous vector fields b , where the authors prove under certain conditions the existence of a unique limiting law on a small time interval which is concentrated on at most two trajectories. The proof of the latter results relies on estimates of mean exit times of X_\cdot^ε with respect to (small) neighbourhoods of isolated singular points of b by means of solutions of an associated boundary value problem. We also mention the papers [17], [18], where the authors use large deviation techniques to study the convergence rate of the laws of X_\cdot^ε for a concrete class of one-dimensional time-homogeneous functions b related to (1.2). In this context it is also worth mentioning the work of [9], which among other things deals with the study of the small noise problem (1.5) based on viscosity solutions of (perturbed) backward Kolmogorov equations in the

scalar case. See also the Malliavin calculus approach in [24] and the article [27] based on local time techniques.

We also remark that extensions of the paper [6] to the case of zero-noise limits of linear transport equations associated with the one-dimensional ODE

$$dX_t = 2\operatorname{sgn}(X_t) |X_t|^\gamma, \gamma \in (0, 1)$$

were analyzed in [3], [4]. See also [14], [23] in the case of zero-noise limits of non-linear PDE's.

Let us now have a look at the small noise problem (1.5) in the multidimensional case. In fact the multidimensional problem is scarcely treated in the literature. Here we shall distinguish between the continuous and discontinuous vector fields:

In the case of bounded and continuous functions b Zhang [30] gives a characterization of the limiting values X^ε of (1.5) by using viscosity solutions of Hamilton-Jacobi-Bellman equations in connection with so-called exit time functions, which partially extends results in [6] to the multidimensional setting.

To the best of our knowledge, the case of discontinuous multidimensional vector fields b has been only examined in the papers of Delarue, Flandoli, Vincenzi [12] and [8]. In the remarkable work [12] the authors study small noise perturbations of the Vlasov-Poisson equation by means of estimates of probabilities for exit times in connection with a zero-noise limit for ODE's in four dimensions. The paper [8] deals with ODE's (1.1) for merely measurable b . However, the concept of solutions to (1.1) in the latter work is in the sense of Filippov, which we don't want to consider in this article.

The objective of our paper is the analysis of zero-noise limits in the case of discontinuous time-inhomogeneous vector fields b in \mathbb{R}^d . More precisely, we aim at considering vector fields b , whose discontinuity points are located in a hyperplane. Our method for the construction of zero-noise limits, which is different from the techniques of the above mentioned authors, is based on estimates of probabilities for exit times of X^ε at discontinuity points in a hyperplane. We comment that our approach extends the one in [6] to the multidimensional case. In contrast to [6] our technique does not require knowledge of the explicit distribution of X^ε . We in fact show that the behavior of the limiting process depends on the normal component of the drift at the upper and lower half-spaces in a neighbourhood of the hyperplane.

2. MAIN RESULTS

Consider an SDE in \mathbb{R}^d with a small noise parameter

$$X^\varepsilon(t) = x^0 + \int_0^t b(s, X^\varepsilon(s)) ds + \varepsilon w(t), \quad (2.1)$$

where $w(t), t \geq 0$, is a Wiener process.

Assume that

$$b(t, x) = \begin{cases} b^+(t, x), & x_d \geq 0 \\ b^-(t, x), & x_d < 0 \end{cases} = b^+(t, x) \mathbb{1}_{x_d \geq 0} + b^-(t, x) \mathbb{1}_{x_d < 0},$$

where $x = (x_1, \dots, x_d) = (\bar{x}, x_d)$, b^\pm are measurable locally bounded functions that satisfy a Lipschitz condition in x on \mathbb{R}^d .

There exists a unique strong solution to SDE (2.1) by Veretennikov's theorem [29].

Note that X^ε spends zero time at the hyper-plane $H := \{x \in \mathbb{R}^d : x_d = 0\}$, so it does not matter how to define the drift coefficient if $x_d = 0$.

Consider a formal limit equation for (2.1)

$$X(t) = x^0 + \int_0^t b(s, X(s))ds. \quad (2.2)$$

Since b is Lipschitz continuous in x outside of H , equation (2.1) has a unique solution up to the moment τ of hitting H ,

$$\tau_H := \inf\{t \geq 0 : X(t) \in H\}.$$

Set

$$\tau_H^{(\varepsilon)} := \inf\{t \geq 0 : X^\varepsilon(t) \in H\}.$$

Theorem 1. *Let $x^0 \notin H$. Then we have the following convergence with probability 1*

$$X^\varepsilon(\cdot \wedge \tau_H) \rightarrow X(\cdot \wedge \tau_H), \quad \varepsilon \rightarrow 0,$$

where X^ε, X are considered as random elements with values in the space of continuous functions $C([0, \infty), \mathbb{R}^d)$ with the topology of uniform convergence on compact sets.

Moreover, if $\mp b^\pm(X(\tau_H)) > 0$, then $\tau_H^{(\varepsilon)} \rightarrow \tau_H, \varepsilon \rightarrow 0$ a.s.

The proof is standard.

Remark 1. If $x^0 \notin H$ and

$$\exists c > 0 \forall t \geq 0 \forall x \in \mathbb{R}_+^d : \operatorname{sgn}(x_d)b_d(t, x) \geq -cx_d,$$

then $\tau = +\infty$ and therefore $X^\varepsilon \rightarrow X, \varepsilon \rightarrow 0$ a.s. in $C([0, \infty), \mathbb{R}^d)$.

For any initial condition x_0 the following result is fulfilled.

Lemma 1. *The sequence of distributions of $\{X^\varepsilon, \varepsilon \in (0, 1)\}$ is weakly relatively compact in $C([0, \infty); \mathbb{R}^d)$.*

For any limit point X of $\{X^\varepsilon\}$ as $\varepsilon \rightarrow 0+$ and for any $t_0 \geq 0$ the following equality holds a.s.

$$X(t) = X(t_0) + \int_{t_0}^t b(s, X(s))ds, \quad t \in [t_0, \tau_{t_0, H}],$$

where $\tau_{t_0, H} = \inf\{t \geq t_0 : X(t) \in H\}$.

The proof of the Lemma is standard.

Remark 2. To prove that the sequence $\{X^\varepsilon, \varepsilon \in (0, 1)\}$ converges in distribution to a process X^0 as $\varepsilon \rightarrow 0+$, it is sufficient to show that for any sequence $\{\varepsilon_k\}$, $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ there exists a subsequence $\{\varepsilon_{k_l}\}$ such that $X^{\varepsilon_{k_l}} \Rightarrow X^0, l \rightarrow \infty$. Since the family $\{X^\varepsilon, \varepsilon \in (0, 1)\}$ is weakly relatively compact, without loss of generality we may initially assume that $\{X^{\varepsilon_k}\}$ is already convergent.

To describe the behavior of processes after the hitting H , let us assume that the initial starting point x_0 belongs to H .

Consider the following cases.

A1 $\exists T > 0 \exists \delta > 0 \exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}^d \setminus H, |x - x^0| \leq \delta : \operatorname{sgn}(x_d)b_d(t, x) \geq c;$

A2₊ $\exists T > 0 \exists \delta > 0 \exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}_+^d, |x - x^0| \leq \delta : b_d^+(t, x) \geq c$ and $\forall x \in H, |x - x^0| \leq \delta : b_d^-(t, x) \geq 0$;

A2₋ $\exists T > 0 \exists \delta > 0 \exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}_-^d, |x - x^0| \leq \delta : b_d^-(t, x) \leq -c$ and $\forall x \in H, |x - x^0| \leq \delta : b_d^+(t, x) \leq 0$;

A3 $\exists T > 0 \exists \delta > 0 \exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}^d \setminus H, |x - x^0| \leq \delta : \operatorname{sgn}(x_d) b_d(t, x) \leq -c$;

A3₊ $\exists T > 0 \exists \delta > 0 \exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}_+^d, |x - x^0| \leq \delta : b_d^+(t, x) \leq -c$ and $\forall x \in H, |x - x^0| \leq \delta : b_d^-(t, x) = 0$;

A3₋ $\exists T > 0 \exists \delta > 0 \exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}_-^d, |x - x^0| \leq \delta : b_d^-(t, x) \geq c$ and $\forall x \in H, |x - x^0| \leq \delta : b_d^+(t, x) = 0$;

A4 $\exists T > 0 \exists \delta > 0 \forall t \in [0, T] \forall x \in H, |x - x^0| \leq \delta : b_d^\pm(t, x) = 0$.

Assume that A1 holds. Then there exist unique (local) solutions to (2.2) that leave H to the positive half-space or negative half-space, respectively. Denote them by $X^+(t)$ and $X^-(t)$,

$$X^\pm(t) = x^0 + \int_0^t b^\pm(s, X^\pm(s)) ds, \quad t \in [0, \tau_H^\pm], \quad (2.3)$$

where

$$\tau_H^\pm = \inf\{t > 0 : X^\pm(t) \in H\}. \quad (2.4)$$

Theorem 2. Assume that A1 is satisfied. The distribution of $X^\varepsilon(\cdot \wedge \tau_H^+ \wedge \tau_H^-)$ converges weakly as $\varepsilon \rightarrow 0$ to the measure

$$p_- \delta_{X^-(\cdot \wedge \tau_H^+ \wedge \tau_H^-)} + p_+ \delta_{X^+(\cdot \wedge \tau_H^+ \wedge \tau_H^-)},$$

where

$$p_\pm = \frac{\pm b_d^\pm(0, x^0)}{b_d^+(0, x^0) - b_d^-(0, x^0)}.$$

Remark 3. Convergence in Theorem 2 could not be a.s. or be convergence in probability. Indeed, assume that a sequence $\{X^\varepsilon\}$ converges a.s. to a process $X^-(t)\mathbb{1}_{\Omega_-} + X^+(t)\mathbb{1}_{\Omega_+}$, where Ω_\pm are disjoint measurable sets, $P(\Omega_\pm) = p_\pm$. It can be shown that $\Omega_\pm \in \mathcal{F}_{0+}$, so their probabilities are either 0 or 1.

Proof. Without loss of generality we will assume that

$$\exists c > 0 \forall t \geq 0 \forall x \in \mathbb{R}^d \setminus H : \operatorname{sgn}(x_d) b^d(t, x) \geq c. \quad (2.5)$$

Let us estimate the time spent by X^ε in the neighborhood of H . By Ito-Tanaka formula, see [28], we have

$$\begin{aligned} |X_d^\varepsilon(t)| &= \int_0^t \operatorname{sgn}(X_d^\varepsilon(s)) b_d(s, X^\varepsilon(s)) ds + \varepsilon \int_0^t \operatorname{sgn}(X_d^\varepsilon(s)) dw_d(s) + L_d^\varepsilon(t) = \\ &= \int_0^t \operatorname{sgn}(X_d^\varepsilon(s)) b_d(s, X^\varepsilon(s)) ds + \varepsilon B^\varepsilon(t) + L_d^\varepsilon(t), \end{aligned}$$

where B^ε is a new Brownian motion, L_d^ε is a non-decreasing, continuous process, $L_d^\varepsilon(0) = 0$.

Therefore, the pair $(|X_d^\varepsilon|, L_d^\varepsilon)$ is a solution of Skorokhod's reflecting problem for the driving process $\xi_\varepsilon(t) = \int_0^t \text{sgn}(X_d^\varepsilon(s))b_d(s, X^\varepsilon(s))ds + \varepsilon B^\varepsilon(t)$. Hence, see for example [25],

$$|X_d^\varepsilon(t)| = \xi_\varepsilon(t) - \min_{s \in [0, t]} \xi_\varepsilon(s).$$

It follows from (2.5) that

$$|X_d^\varepsilon(t)| \geq (ct + \varepsilon B^\varepsilon(t)) - \min_{s \in [0, t]} (cs + \varepsilon B^\varepsilon(s)).$$

Denote $\sigma_{H_\delta}^\varepsilon := \inf\{t \geq 0 : |X_d^\varepsilon(t)| \geq \delta\}$.

Therefore

$$P(|X_d^\varepsilon(t)| \geq \delta, t \geq 2\delta/c) \rightarrow 1, \varepsilon \rightarrow 0+, \quad (2.6)$$

$$P(\sigma_{H_\delta}^\varepsilon > 2\delta/c) \rightarrow 0, \varepsilon \rightarrow 0+. \quad (2.7)$$

Moreover if $\delta < 1 \wedge c/2M$, where $M = \max_{t \in [0, 1], |x - x^0| \leq 1} |\bar{b}(t, x)|$, then

$$P(|\bar{X}(\sigma_{H_\delta}^\varepsilon) - \bar{x}^0| > 2\delta M/c) \rightarrow 0, \varepsilon \rightarrow 0+. \quad (2.8)$$

It follows from (2.6), Lemma 1, and assumption (2.5) that for any limit point X we have with probability 1:

$$X(t) = x^0 + \mathbb{1}_{\Omega_+} \int_0^t b^+(s, X(s))ds + \mathbb{1}_{\Omega_-} \int_0^t b^-(s, X(s))ds,$$

where $\Omega_+ = \{\omega : X(t) > 0 \text{ for all } t > 0\}$, $\Omega_- = \{\omega : X(t) < 0 \text{ for all } t > 0\}$.

Notice that if (2.5) is true, then

$$P(\Omega_+ \cup \Omega_-) = 1, P(\Omega_- \Delta \{X_d(\sigma_{H_\delta}) = -\delta\}) = 0, \text{ and } P(\Omega_+ \Delta \{X_d(\sigma_{H_\delta}) = \delta\}) = 0 \quad (2.9)$$

for any $\delta > 0$.

Let $\delta > 0$, $\beta > 0$ be sufficiently small fixed numbers and α be such that for all $t \in [0, \delta]$, $x \in [x^0 - \beta, x^0 + \beta]$:

$$\begin{aligned} 0 &< b_d^+(t, x) - \alpha < b_d^+(0, x_0) < b_d^+(t, x) + \alpha, \\ 0 &< -b_d^-(t, x) - \alpha < -b_d^-(0, x_0) < -b_d^-(t, x) + \alpha. \end{aligned}$$

Define processes:

$$X^{\varepsilon, \pm\alpha}(t) = x^0 + \int_0^t (b_d^+(0, x_0)\mathbb{1}_{X^{\varepsilon, +\alpha}(s) \geq 0} + b_d^-(0, x_0)\mathbb{1}_{X^{\varepsilon, +\alpha}(s) < 0} \pm \alpha)ds + \varepsilon w(t), t \geq 0.$$

By comparison theorem, see [19], we have a.s.

$$X^{\varepsilon, -\alpha}(t) \leq X^d(t) \leq X^{\varepsilon, +\alpha}(t)$$

for all $t \in [0, \sigma_{H_\delta}^\varepsilon \wedge \inf\{s : |\bar{X}(s) - \bar{x}^0| > \beta\}]$.

The processes $X^{\varepsilon, \pm\alpha}$ are one-dimensional homogeneous diffusions. So, see [16, 19],

$$P(X_d^{\varepsilon, \pm\alpha}(\sigma_{H_\delta}^{\pm\alpha}) = \delta) =$$

$$\frac{(\pm\alpha - b_d^-(0, x_0))^{-1}(1 - \exp(2\delta\varepsilon^{-2}(b_d^-(0, x_0) \pm \alpha)))}{(\pm\alpha - b_d^-(0, x_0))^{-1}(1 - \exp(2\delta\varepsilon^{-2}(b_d^-(0, x_0) \pm \alpha))) + (\pm\alpha + b_d^+(0, x_0))^{-1}(1 - \exp(2\delta\varepsilon^{-2}(b_d^+(0, x_0) \pm \alpha)))},$$

where $\sigma_{H_\delta}^{\pm\alpha} := \inf\{t \geq 0 : |X^{\varepsilon, \pm\alpha}(t)| \geq \delta\}$.

This and (2.9) conclude the proof. \square

Theorem 3. Let $x^0 \in H$ and $A2_+$ or $A2_-$ be satisfied. Then

$$X^\varepsilon(\cdot \wedge \tau_H^\pm) \rightarrow X^\pm(\cdot \wedge \tau_H^\pm), \varepsilon \rightarrow 0, \text{ a.s.},$$

where the sign $+$ or $-$ is selected accordingly to the sign in condition $A2$, τ_H^\pm is defined in (2.4).

Remark 4. If condition $A2_\mp$ is satisfied at the point $X^\pm(\tau_H^\pm)$, then we may define a moment τ_\pm^2 similarly to $\tau_\pm^1 := \tau_H^\pm$ and obtain the similar convergence of processes $\{X^\varepsilon\}$ on $[\tau_\pm^1, \tau_\pm^2]$, and so on. Note that if $A2_+$ is satisfied in x^0 , then $A2_+$ cannot be true in $X^+(\tau_+^1)$.

Proof. Assume that $A2_+$ is satisfied. Without loss of generality we may assume that

$$\exists c > 0 \forall t \in [0, T] \forall x \in \mathbb{R}_+^d : b_+^d(t, x) \geq c$$

and

$$\forall t \in [0, T] \forall x \in H : b_-^d(t, x) \geq 0.$$

Observe that

$$\begin{aligned} X^\varepsilon(t) &= x^0 + \int_0^t (\mathbb{1}_{\{X^\varepsilon(s) > 0\}} b^+(s, X^\varepsilon(s)) + \mathbb{1}_{\{X^\varepsilon(s) < 0\}} b^-(s, X^\varepsilon(s))) ds + \varepsilon w(t) = \\ &= x^0 + \int_0^t b^+(s, X^\varepsilon(s)) ds + \int_0^t \mathbb{1}_{\{X^\varepsilon(s) < 0\}} (b^-(s, X^\varepsilon(s)) - b^+(s, X^\varepsilon(s))) ds + \varepsilon w(t). \end{aligned}$$

Lemma 1 and Remark 2 yield that to prove the Theorem it suffices to verify that any limit point X of $\{X_\varepsilon\}$ as $\varepsilon \rightarrow 0$ is such that

$$\int_0^T \mathbb{1}_{\{X_d(s) \leq 0\}} ds = 0 \text{ a.s.} \quad (2.10)$$

Function b^- is Lipschitz continuous. So $b_d^-(t, x) \geq Lx_d, x \in \mathbb{R}_-^d$.

For any $\alpha > 0$ set

$$X_d^{\varepsilon, \alpha}(t) = \int_0^t (\mathbb{1}_{\{X_d^{\varepsilon, \alpha}(s) < 0\}} (LX_d^{\varepsilon, \alpha}(s) - \alpha) + \mathbb{1}_{\{X_d^{\varepsilon, \alpha}(s) \geq 0\}} c) ds + \varepsilon w(t).$$

By comparison theorem $X_d^\varepsilon(t) \geq X_d^{\varepsilon, \alpha}(t), t \in [0, T]$ a.s.

Therefore for any limit point X of $\{X_\varepsilon\}$ is such that

$$P(X_d(t) \geq \delta, t \geq 2c/\delta) \geq \limsup_{\varepsilon \rightarrow 0} P(X_d^\varepsilon(t) \geq \delta, t \geq 2c/\delta)$$

$$\geq \limsup_{\varepsilon \rightarrow 0} P(X_d^{\varepsilon, \alpha}(t) \geq \delta, t \geq 2c/\delta) \geq P(X_d^\alpha(t) \geq \delta, t \geq 3c/2\delta) = P(X_d^\alpha(t) > 0, t > 0) = \frac{c}{c + \alpha},$$

where X_d^α is a limit of $\{X_d^{\varepsilon, \alpha}\}$ as $\varepsilon \rightarrow 0$.

Since α is arbitrary, we have (2.10). \square

Lemma 2. Assume that $x^0 \in H$, and condition $A3$ or $A4$ is satisfied. Let X^0 be any (weak) limit point for $\{X^\varepsilon\}$. Denote

$$\sigma_\delta = \sigma_\delta(X^0) := \inf\{t \geq 0 : |X^0(t) - x^0| \geq \delta\}, \quad (2.11)$$

where $\delta > 0$ is a parameter from conditions $A3, A4$.

Then

$$P(X^0(t) \in H, t \in [0, \sigma_\delta \wedge T]) = 1.$$

Proof. We prove Lemma if the global condition A4 is satisfied:

$$\forall t \geq 0 \forall x \in H : b_{\pm}^d(t, x) = 0.$$

All other cases are considered similarly.

Assume that $X^{\varepsilon_k} \Rightarrow X^0, k \rightarrow \infty$. By Skorokhod's theorem on a single probability space we may assume that the convergence is a.s.:

$$\forall T > 0 : \sup_{t \in [0, T]} |X^{\varepsilon_k}(t) - X^0(t)| \rightarrow 0, k \rightarrow \infty. \quad (2.12)$$

Assume that for some ω and t_1 we have $X_d^0(t_1) > 0$. Denote by $t_0 \in [0, t_1]$ the last visit of H by X^0 , i.e.,

$$t_0 = \sup\{s \in [0, t_1] : X_d^0(s) = 0\}.$$

Due to (2.12) we have

$$X_d^0(t) = \int_{t_0}^{t_1} b_d^+(s, X^0(s)) ds, \quad t \in [t_0, t_1].$$

Since $X_d^0(t_0) = 0, X_d^0(t) \geq 0, t \in [t_0, t_1]$, by assumption A4 and Lipschitz property we have

$$X_d^0(t) \leq L \int_{t_0}^{t_1} X_d^0(s) ds, \quad t \in [t_0, t_1].$$

The application of Gronwall's lemma completes the proof. \square

Theorem 4. Assume that $x^0 = (\bar{x}^0, 0) \in H$ and A3 (or A3 $_{\pm}$) is satisfied, functions b^{\pm} are continuous in (t, x) . Denote

$$p^{\pm}(s, \bar{x}) = \frac{b_d^{\pm}(s, (\bar{x}, 0))}{b_d^-(s, (\bar{x}, 0)) + b_d^+(s, (\bar{x}, 0))}.$$

Let $\bar{X}(t) \in \mathbb{R}^{d-1}$ be a solution of the ordinary differential equation

$$\begin{aligned} \bar{X}^0(t) = \bar{x}^0 + \int_0^t & (\bar{b}^+(s, (\bar{X}^0(s), 0))p^-(s, \bar{X}^0(s)) + \\ & + \bar{b}^-(s, (\bar{X}^0(s), 0))p^+(s, \bar{X}^0(s))) ds, \quad t \in [0, \sigma_{\delta}(X^0) \wedge T]. \end{aligned} \quad (2.13)$$

Then

$$X^{\varepsilon}(\cdot \wedge \sigma_{\delta}(X^0) \wedge T) \xrightarrow{P} (\bar{X}^0(\cdot \wedge \sigma_{\delta}(X^0) \wedge T), 0), \varepsilon \rightarrow 0,$$

Remark 5. Coefficients of the equation (2.13) are continuous in (s, \bar{x}) and Lipschitz in \bar{x} until a solution exits δ -neighborhood of x^0 or $t > T$. Thus there exists a unique solution to (2.13).

Proof. We only consider the case

$$\exists c > 0 \forall t \geq 0 \forall x \in \mathbb{R}^d \setminus H : \operatorname{sgn}(x_d) b^d(t, x) \leq -c.$$

Let $\{X^{\varepsilon_k}\}$ be any weakly convergent subsequence (see Lemma 1). By Skorokhod's theorem on a single probability space we may assume that (2.12) holds with probability 1 and also

$$\forall T > 0 \sup_{t \in [0, T]} \left| \int_0^t b_d(s, X^{\varepsilon_k}(s)) ds \right| \rightarrow 0, k \rightarrow \infty. \quad (2.14)$$

Lemma 3. *Let ω be such that (2.12) and (2.14) are satisfied. Then*

$$\forall t \geq 0 : \lim_{k \rightarrow \infty} \int_0^t \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds = \int_0^t \frac{b_d^-(s, X^0(s))}{b_d^-(s, X^0(s)) + b_d^+(s, X^0(s))} ds. \quad (2.15)$$

Remark 6. Since all functions under the integral signs in (2.15) are bounded, convergence in Lemma 3 is locally uniform.

Proof. Let s_0 be arbitrary and $\Delta = [t_1, t_2]$ is such that $s_0 \in \Delta$. Then

$$\begin{aligned} \left| \int_{\Delta} b_d(s, X^{\varepsilon_k}(s)) ds \right| &= \int_{\Delta} b_d^-(s, X^{\varepsilon_k}(s)) \mathbb{1}_{X^{\varepsilon_k}(s) < 0} ds - \int_{\Delta} b_d^+(s, X^{\varepsilon_k}(s)) \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds = \\ &= b_d^-(s_0, X^{\varepsilon_k}(s_0)) \int_{\Delta} \mathbb{1}_{X^{\varepsilon_k}(s) < 0} ds - b_d^+(s_0, X^{\varepsilon_k}(s_0)) \int_{\Delta} \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds + \\ &+ \int_{\Delta} (b_d^-(s, X^{\varepsilon_k}(s)) - b_d^-(s_0, X^{\varepsilon_k}(s_0))) \mathbb{1}_{X^{\varepsilon_k}(s) < 0} ds - \int_{\Delta} (b_d^+(s, X^{\varepsilon_k}(s)) - b_d^+(s_0, X^{\varepsilon_k}(s_0))) \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds. \end{aligned}$$

It follows from the last equality and (2.12) that

$$\begin{aligned} \left| \int_{\Delta} b_d(s, X^{\varepsilon_k}(s)) ds \right| &\geq \\ &\geq b_d^-(s_0, X^0(s_0)) \int_{\Delta} \mathbb{1}_{X^{\varepsilon_k}(s) < 0} ds - b_d^+(s_0, X^0(s_0)) \int_{\Delta} \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds + o_{\varepsilon_k}(1) |\Delta| + o(|\Delta|) = \\ &= |\Delta| \left(\frac{b_d^-(s_0, X^0(s_0))}{b_d^-(s_0, X^0(s_0)) + b_d^+(s_0, X^0(s_0))} - |\Delta|^{-1} \int_{\Delta} \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds + o_{\varepsilon_k}(1) + o(|\Delta|)/|\Delta| \right), \end{aligned} \quad (2.16)$$

where $o_{\varepsilon_k}(1)$ is independent of Δ , $o_{\varepsilon_k}(1) \rightarrow 0$, $k \rightarrow \infty$, and $o(\Delta)$ is independent of k .

If (2.15) is not true, then there exists a point s_0 , a subsequence $\{\varepsilon_{k_l}\}$, and a sequence of intervals $\{\Delta_n\}$, $\Delta_{n+1} \subset \Delta_n$, $s_0 \in \Delta_n$, $n \geq 1$, $\lim_{n \rightarrow \infty} |\Delta_n| = 0$ such that

$$\liminf_{n \rightarrow \infty} \liminf_{l \rightarrow \infty} \left| |\Delta_n|^{-1} \int_{\Delta_n} \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds - \frac{b_d^-(s_0, X^0(s_0))}{b_d^-(s_0, X^0(s_0)) + b_d^+(s_0, X^0(s_0))} \right| > 0.$$

This contradicts (2.16). \square

The following result is well known.

Lemma 4. *Assume that $\{f_n\}$, $\{l_n\} \subset C([0, T])$ are uniformly convergent sequences of continuous functions*

$$f_n \rightarrow f_0 \text{ and } l_n \rightarrow l_0, \quad n \rightarrow \infty,$$

and each function l_n is non-decreasing.

Then we have the uniform convergence of the integrals

$$\sup_{t \in [0, T]} \left| \int_0^t f_n(s) dl_n(s) - \int_0^t f_0(s) dl_0(s) \right| \rightarrow 0, \quad n \rightarrow \infty.$$

The proof of weak convergence $X^{\varepsilon} \Rightarrow X^0$ follows from Lemmas 3, 4, Remark 2, and (2.12) if we set (recall that X^{ε_k} are copies) $f_n^{\pm}(t) := b^{\pm}(t, X^{\varepsilon_n}(t))$, $l_n^{\pm}(t) := \int_0^t \mathbb{1}_{X^{\varepsilon_k}(s) \geq 0} ds$, $l_0^{\pm}(t) := \int_0^t \frac{\pm b_d^{\mp}(s, X^0(s))}{b_d^-(s, X^0(s)) + b_d^+(s, X^0(s))} ds$. Since X_0 is non-random, weak convergence implies convergence in probability. \square

Consider the case A4.

Assume that the limits exist

$$c^\pm(t, \bar{x}) := \lim_{x_d \rightarrow 0^\pm} \frac{b_d^\pm(t, x)}{x_d}, t \in [0, T], |\bar{x} - \bar{x}^0| < \delta.$$

Since $b_d^\pm(t, x) = 0, t \in [0, T], |x - x^0| < \delta$, and Lipschitz in x , functions c_d^\pm are bounded for $t \in [0, T], |\bar{x} - \bar{x}^0| < \delta$.

Consider the system

$$\begin{cases} \bar{X}(t) = \bar{x}^0 + \int_0^t (\bar{b}^+(s, (\bar{X}(s), 0))\mathbb{1}_{Y(s) \geq 0} + \bar{b}^-(s, (\bar{X}(s), 0))\mathbb{1}_{Y(s) < 0}) ds, \\ Y(t) = \int_0^t \left(c^+(s, \bar{X}(s))\mathbb{1}_{Y(s) \geq 0} + c^-(s, \bar{X}(s))\mathbb{1}_{Y(s) < 0} \right) Y(s) ds + w_d(t). \end{cases} \quad (2.17)$$

Lemma 5. 1) There exists a unique weak solution to (2.17) defined up to the moment $\sigma_\delta(\bar{X}) \wedge T$, where $\sigma_\delta(\bar{X}) := \inf\{t \geq 0 : |\bar{X}(t) - \bar{x}^0| \geq \delta\}$. 2) (a) There exists a unique unique strong solution to (2.17) defined up to the moment $\sigma_\delta(\bar{X}) \wedge T$, if the functions $c^\pm(s, \bar{x}) = c^\pm(s)$ are independent of \bar{x} .

b) The system (2.17) has a unique maximal solution, if e.g. the functions $\bar{b}^\pm(s, (\bar{x}, 0)), c^\pm(s, \bar{x})$ belong to $C^{3,3}([0, T] \times \mathbb{R}^{d-1})$.

Proof. Proof of the weak existence and uniqueness follows from the Girsanov theorem. Proof in the case 2a) is obvious because all coefficients are Lipschitz continuous in the spatial variable.

As for the proof of case 2b), see Theorem 3.2 in [22]. \square

Theorem 5. Assume that $x^0 \in H$, functions c^\pm are continuous in (t, x) , and assumption A4 is satisfied.

Then we have the weak convergence

$$\begin{aligned} (\bar{X}^\varepsilon(\cdot \wedge \sigma_\delta(\bar{X}^\varepsilon) \wedge T), \varepsilon^{-1} X_d^\varepsilon(\cdot \wedge \sigma_\delta(\bar{X}^\varepsilon) \wedge T)) &\rightarrow \\ &\rightarrow (\bar{X}(\cdot \wedge \sigma_\delta(\bar{X}) \wedge T), Y(\cdot \wedge \sigma_\delta(\bar{X}) \wedge T)), \quad \varepsilon \rightarrow 0, \end{aligned}$$

where (\bar{X}, Y) is a solution of (2.17).

In particular $X^\varepsilon(\cdot \wedge \sigma_\delta(\bar{X}^\varepsilon) \wedge T) \rightarrow (\bar{X}(\cdot \wedge \sigma_\delta(\bar{X}) \wedge T), 0)$ weakly.

Moreover, if there exists a strong solution to (2.17), then not only weak convergence holds but also convergence in probability.

Remark 7. Weak uniqueness and strong existence yield uniqueness of the strong solution, see reasoning of [10].

Proof. For simplicity we assume that

$$\forall t \geq 0 \forall x \in H : b_\pm^d(t, x) = 0.$$

Set $Y^\varepsilon(t) := X_d^\varepsilon(t)/\varepsilon$. Then

$$\begin{cases} \bar{X}^\varepsilon(t) = \bar{x}^0 + \int_0^t (\bar{b}^+(s, (\bar{X}^\varepsilon(s), Y^\varepsilon(s)))\mathbb{1}_{Y^\varepsilon(s) \geq 0} + \bar{b}^-(s, (\bar{X}^\varepsilon(s), Y^\varepsilon(s)))\mathbb{1}_{Y^\varepsilon(s) < 0}) ds + \varepsilon \bar{w}(t), \\ Y^\varepsilon(t) = \int_0^t \left(\frac{b^+(s, (\bar{X}^\varepsilon(s), \varepsilon Y^\varepsilon(s)))}{\varepsilon Y^\varepsilon(s)} \mathbb{1}_{Y^\varepsilon(s) \geq 0} + \frac{b^-(s, (\bar{X}^\varepsilon(s), \varepsilon Y^\varepsilon(s)))}{\varepsilon Y^\varepsilon(s)} \mathbb{1}_{Y^\varepsilon(s) < 0} \right) Y^\varepsilon(s) ds + w_d(t). \end{cases}$$

Remark 8. $\int_0^\infty \mathbb{1}_{Y^\varepsilon(s)=0} ds = 0$ a.s.

It can be readily shown that a family $\{(\bar{X}^\varepsilon, Y^\varepsilon), \varepsilon \in (0, 1)\}$ is weakly relatively compact in $C([0, \infty); \mathbb{R}^d)$. Let $\{(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n})\}$ be a convergent subsequence, where $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. By theorem on a single probability space, there is a sequence of copies $(\tilde{X}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n}, \tilde{w}^{\varepsilon_n}) \stackrel{d}{=} (\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n}, w)$ such that

$$\begin{cases} \tilde{X}^{\varepsilon_n}(t) = \bar{x}^0 + \int_0^t \left(\bar{b}^+(s, (\tilde{X}^{\varepsilon_n}(s), \tilde{Y}^{\varepsilon_n}(s))) \mathbb{1}_{\tilde{Y}^{\varepsilon_n}(s) \geq 0} + \bar{b}^-(s, (\tilde{X}^{\varepsilon_n}(s), \tilde{Y}^{\varepsilon_n}(s))) \mathbb{1}_{\tilde{Y}^{\varepsilon_n}(s) < 0} \right) ds + \varepsilon_n \tilde{w}^{\varepsilon_n}(t), \\ \tilde{Y}^{\varepsilon_n}(t) = \int_0^t \left(\frac{b^+(s, (\tilde{X}^{\varepsilon_n}(s), \varepsilon_n \tilde{Y}^{\varepsilon_n}(s)))}{\varepsilon_n \tilde{Y}^{\varepsilon_n}(s)} \mathbb{1}_{\tilde{Y}^{\varepsilon_n}(s) \geq 0} + \frac{b^-(s, (\tilde{X}^{\varepsilon_n}(s), \varepsilon_n \tilde{Y}^{\varepsilon_n}(s)))}{\varepsilon_n \tilde{Y}^{\varepsilon_n}(s)} \mathbb{1}_{\tilde{Y}^{\varepsilon_n}(s) < 0} \right) \tilde{Y}^{\varepsilon_n}(s) ds + \tilde{w}_d^{\varepsilon_n}(t). \end{cases}$$

and

$$(\tilde{X}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n}, \tilde{w}^{\varepsilon_n}) \rightarrow (\tilde{X}, \tilde{Y}, \tilde{w})$$

almost surely.

Observe that

$$\lim_{n \rightarrow \infty} \int_0^t \left(\frac{b^+(s, (\tilde{X}^{\varepsilon_n}(s), \varepsilon_n \tilde{Y}^{\varepsilon_n}(s)))}{\varepsilon_n \tilde{Y}^{\varepsilon_n}(s)} \mathbb{1}_{\tilde{Y}^{\varepsilon_n}(s) \geq 0} + \frac{b^-(s, (\tilde{X}^{\varepsilon_n}(s), \varepsilon_n \tilde{Y}^{\varepsilon_n}(s)))}{\varepsilon_n \tilde{Y}^{\varepsilon_n}(s)} \mathbb{1}_{\tilde{Y}^{\varepsilon_n}(s) < 0} \right) \tilde{Y}^{\varepsilon_n}(s) ds$$

is of the form $\int_0^t \xi(s) ds$, where $\xi(t)$ is independent of σ -algebra generated by $(\tilde{w}_d(s) - \tilde{w}_d(t)), s \geq t$. So $\int_0^\infty \mathbb{1}_{\tilde{Y}(s)=0} ds = 0$ a.s.

It follows from the Lebesgue dominated convergence theorem that the limit process (\tilde{X}, \tilde{Y}) is a solution of (2.17) with \tilde{w}_d in place of w_d . Since $\{(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n})\}$ was arbitrary convergent subsequence, the proof of the Theorem follows from the weak uniqueness of the solution to (2.17).

If there exists a unique strong solution to (2.17), consider then a.s. convergent sequence of copies of $(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n}, w, \bar{X}, Y)$:

$$(\tilde{X}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n}, \tilde{w}^{\varepsilon_n}, \hat{X}^{\varepsilon_n}, \hat{Y}^{\varepsilon_n}) \rightarrow (\tilde{X}, \tilde{Y}, \tilde{w}, \hat{X}, \hat{Y}).$$

It can be seen that the limit processes (\tilde{X}, \tilde{Y}) and (\hat{X}, \hat{Y}) satisfy the same equation with the same Wiener process \tilde{w} . It follows from uniqueness of the strong solution that $(\tilde{X}, \tilde{Y}) = (\hat{X}, \hat{Y})$ a.s. So $(\tilde{X}^{\varepsilon_n}, \tilde{Y}^{\varepsilon_n}) - (\hat{X}^{\varepsilon_n}, \hat{Y}^{\varepsilon_n}) \rightarrow 0$ a.s. Therefore $(\bar{X}^{\varepsilon_n}, Y^{\varepsilon_n}) - (\bar{X}, Y) \rightarrow 0$ in probability. \square

Example 1. A limit of X^ε may be non-Markov in case A4. Indeed, assume that $\bar{b}^\pm(t, x) = \bar{b}^\pm = \text{const}$, $\bar{b}^+ \neq \bar{b}^-$, $b_d^\pm(t, x) = 0$, and $x^0 \in H$.

It follows from Theorem 5 that

$$X^\varepsilon(\cdot) \rightarrow (\bar{X}^0(\cdot), 0), \varepsilon \rightarrow 0,$$

where

$$\bar{X}^0(t) := x^0 + \bar{b}^+ l^+(t) + \bar{b}^- l^-(t),$$

$l^\pm(t)$ is the time spent by $w_d(s), s \in [0, t]$, in the positive half-space and negative half-space, respectively.

The process $\bar{X}^0(t), t \geq 0$, is not a Markov process.

REFERENCES

- [1] Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* 158(2):227-260 (2004).
- [2] Ambrosio, L., Crippa, G.: Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields. *UMI-Lecture Notes* (2006).
- [3] Attanasio, F., Flandoli, F.: Zero-noise solutions of linear transport equations without uniqueness: an example. *C.R. Acad. Sci. Paris, Ser. I*, 347, 753-756 (2009).
- [4] Attanasio, S.: L'equazione del trasporto deterministica e stocastica. *Tesi di Laurea in Matematica*, Università di Pisa (2010).
- [5] Bafico, R.: On the convergence of the weak solutions of stochastic differential equations when the noise intensity goes to zero. *Bollettino UMI* (5), 308-324 (1980).
- [6] R. Bafico, P. Baldi, *Small random perturbations of Peano phenomena*, *Stochastics* 6 (1982), n. 3, 279-292.
- [7] Bafico, R., Baldi, P.: Small Random Perturbations of Peano Phenomena. *Stochastics*. Vol. 6, pp. 279-292 (1982).
- [8] Buckdahn, R., Ouknine, Y., Quincampoix, M.: On limiting values of stochastic differential equations with small noise intensity tending to zero. *Bull. Sci. Math.* 133:229-237 (2009).
- [9] Borkar, V.S., Suresh Kumar, K.: A new Markov selection procedure for degenerate diffusions. *J. Theoret. Probab.* 23, no. 3, 729-747 (2010).
- [10] Cherny, A. S. (On the strong and weak solutions of stochastic differential equations governing Bessel processes. *Stochastics: An International Journal of Probability and Stochastic Processes*, 2000), 70(3-4), 213-219.
- [11] F. Delarue, F. Flandoli, *The transition point in the zero noise limit for a 1D Peano example*, *Discrete Contin. Dyn. Syst.*, 34 (2014), n. 10, 4071-4083.
- [12] Delarue, F., Flandoli, F., Vincenzi, D.: Noise prevents collapse of Vlasov-Poisson point charges. *Communications on Pure and Applied Math.*, Vol. 67, Issue 10, 1700-1736 (2014).
- [13] Di Perna, R.J., Lions, P.-L.: Ordinary differential equations, transport theory, and Sobolev spaces. *Invent. Math.* 98(3):511-547 (1989).
- [14] Dirr, N., Luckhaus, S., Novaga, M.: A stochastic selection principle in case of fattening for curvature flow. *Calc. Var. Part. Diff. Eq.* 13, No. 4, 405-425 (2001).
- [15] Flandoli, F.: Remarks on uniqueness and strong solutions to deterministic and stochastic differential equations. *Metrika*, 69:101-123 (2009).
- [16] I.I.Gikhman, A.V.Skorokhod. *Stochastic differential equations*. (Russian) *Naukova Dumka*, Kiev. 1968. - 354 p.; English translation - *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band 72. - Springer-Verlag, New York; Heidelberg. 1972. - viii+354 p.
- [17] Gradinaru, M., Herrmann, S., Roynette, B.: A singular large deviations phenomenon. *Ann. Inst. H. Poincaré Probab. Statist.* 37, (5), 555-580 (2001).
- [18] Herrmann, S.: Phénomène de Peano et grandes déviations. *C.R. Acad. Sci. Paris Sér. I Math.* 332 no. 11, 1019-1024 (2001).
- [19] Ikeda, N., Watanabe, S. *Stochastic differential equations and diffusion processes*. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981. xiv+464 pp.
- [20] I.G. Krykun, S. Ya. Makhno, *The Peano phenomenon for Ito equations*, *Journal of Mathematical Sciences*, Vol. 192, No. 4, July, 2013
- [21] Krylov, N.V.: The selection of a Markov process from a Markov system of processes (Russian). *Izv. Akad. Nauk. SSSR Ser. Mat.* 37:691-708 (1973).
- [22] Leobacher, G., Thonhauser, S., Szölgyenyi, M.: On the existence of solutions of a class of SDE's with discontinuous drift and singular diffusion. [arXiv:1311.6226v4 \[math.PR\]](https://arxiv.org/abs/1311.6226v4) 6Feb 2015.
- [23] Mariani, M.: Large deviation principles for stochastic scalar conservation laws. *Prob. Th. and Rel. Fields* 147, No. 3-4, 607-648 (2010).

- [24] Menoukeu-Pamen, O., Meyer-Brandis, T., Proske, F.: A Gel'fand triple approach to the small noise problem for discontinuous ODE's. Preprint server, University of Oslo, No.25, ISSN, 0806-2439 (2010).
- [25] Pilipenko A. An introduction to stochastic differential equations with reflection / Andrey Pilipenko. ÖPotsdam : Universitätsverlag, 2014. Öäix, 75 S. graph. Darst. (Lectures in pure and applied mathematics 1); ISSN (print) 2199-4951; ISSN (online) 2199-496X ISBN 978-3-86956-297-1
- [26] Stampacchia, G.: Le trasformazioni funzionali che presentano il fenomeno di Peano. Atti Accad Nazz Lincei Rend Cl Sci Fis Mat Nat 7:80-84 (1949).
- [27] D. Trevisan, *Zero noise limits using local times*, Electron. Commun. Probab. 18 (2013), no. 31, 7 pp.
- [28] Revuz, D., Yor, M. Continuous martingales and Brownian motion. Third edition. Springer-Verlag, Berlin, 1999. xiv+602 pp.
- [29] Veretennikov A. *On strong solutions and explicit formulas for solutions of stochastic integral equations*. Math. USSR Sborn, 39(3):387Öt03, 1981.
- [30] Zhang, L.: Random perturbation of some multidimensional non-Lipschitz ordinary differential equations. arXiv:1202.4131v5 [math.PR], 2 Apr 2013.

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, TERESHCHENKIVSKA STR. 3, 01601, KIEV, UKRAINE

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, N-316 OSLO, NORWAY